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# A general linear theory of laminated composite shells featuring interlaminar bonding imperfections

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Dedicated to Professors Charles W. Bert and Jack R. Vinson on the occasion of their 70th birthday.

**Abstract**

This paper is devoted to the foundation of a general linear theory of laminated composite anisotropic shells of arbitrary shape and curvature, in which the effect of the interfacial damage induced by the imperfect bonding between the constituent laminae is incorporated. In this context, the imperfect interface conditions are described in terms of linear relations between the interface tractions in the normal and tangential directions, and the respective displacement jumps. In addition to the effects of imperfectly bonded interfaces, the theory incorporates the effects of transverse shear and transverse normal strain, the dynamic effects, as well as the anisotropy of constituent material layers. Due to its general character, this theory can contribute to a more reliable prediction in the linear range of the load carrying capacitiy and failure of laminated composite shell structures featuring imperfectly bonded interfaces. 

## 1. Introduction

The requirements for higher strength-to-weight and stiffness-to-weight ratios, better corrosion resistance, longer fatigue life, greater stealth characteristics over metals as well as the directionality properties, have resulted in increasing demand of laminated composite structures in many challenging applications. Among these, there are the future supersonic/hypersonic launch and reusable flight vehicle operating in space, advanced propulsion systems, etc. In spite of their increased flexibility, the structure of next generation of high speed flight vehicles, has to be able to operate in complex environmental conditions and feature an expanded operational envelope. A problem of crucial importance toward the rational design of these structure consists of the possibility to accurately determine their load carrying capacity. The exhaustive use of the load carrying capacity of such structures can dramatically contribute to the increase, without weight penalties, of the performance of such vehicles.

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As a result of the ongoing trend consisting of the incorporation of advanced composite materials in their construction, and for a reliable evaluation of their load carrying capacity, a careful assessment of the implications played by a number of non-classical effects is required. In this sense, a thorough discussion of the state of the art of multilayered composite shells and of the issues which have to be solved still to obtain reliable models was accomplished by Noor and Burton (1990).

One of the important factors which has to be addressed is related to the transverse shear and transverse normal effects.

An additional factor which becomes relevant in the context of *multilayered* plates and shells composed of advanced composite materials is of a modeling nature. This is related to the non-fulfillment of the continuity requirement of shear and transverse normal tractions across the perfectly bonded interfaces. As was revealed in different contexts (Di Sciuva, 1987, 1994; Soldatos and Timarci, 1993; Timarci and Soldatos, 1995; Di Sciuva et al., 1997; Librescu and Lin, 1999; He, 1994; Ossadzow et al., 1995; Carrera, 1999a,b,c), the violation of this requirement can result in unavoidable errors in the evaluation of the load carrying capacity of laminated composite structures.

Moreover, as a result of manufacturing processes and/or operating conditions, interfacial bonding damages between the constituent laminae of the composite structures, resulting in debonding or imperfectly bonded interfaces, can occur. These flaws result in stiffness degradation with detrimental implications upon the response behavior of laminated composite structures, in general, and on their load carrying capacity, in particular. In this paper, the effect of damage due to the imperfect bonding between the constituent laminae is incorporated. In this context, various cases of imperfectly bonded interfaces, including, as special cases, complete debonding, slip type interlaminar imperfection, and perfect bonding will be considered in a unified way.

Herein, a rather general linear theory of anisotropic laminated composite shell structures characterized by a general lay-up configuration, featuring damaged bonded interfaces and incorporating the transverse shear and transverse normal effects will be developed. Employment of a linear relationship between the interface tractions and displacement jumps across the imperfect interfaces, and using the Hamilton variational principle of the 3-D elastodynamics in conjunction with the postulated displacement field, results in the equations of motion and boundary conditions. Special cases of the obtained equations are displayed and some comments on the alternative ways enabling one to express the static and kinematic 2-D quantities are also included.

In spite of its evident importance, as far as the authors of the present paper are aware, the theory of laminated composite shells has not yet been approached in a so general context.

This paper represents a continuation and development of a number of results previously obtained by Librescu and Schmidt (1991), where the case of perfectly bonded interfaces was considered, as well as of the papers by Schmidt and Librescu (1996, 1999), where for laminated composite plates and shells, respectively, a special case of interlaminae flaws was considered.

## 2. Preliminaries

Consider a composite laminated shell consisting of a finite number of linearly elastic anisotropic layers, each of these exhibiting different physico-mechanical properties. It is assumed that the interfaces between the contiguous layers may feature imperfect bondings which can affect the degree of adhesion at the interfaces in the normal and tangential directions.

The thickness of the $k$th constituent layer and of the entire shell are denoted by $h_{(k)}$ $(k = 1, 2, \ldots, N)$ and $h$, respectively, where $N$ denotes the total number of constituent layers.

For the sake of convenience, the undeformed mid-surface of the bottom layer is selected as the reference surface $\sigma$ (Fig. 1). The points of the 3-D shell structure are referred to an arbitrary curvilinear coordinate

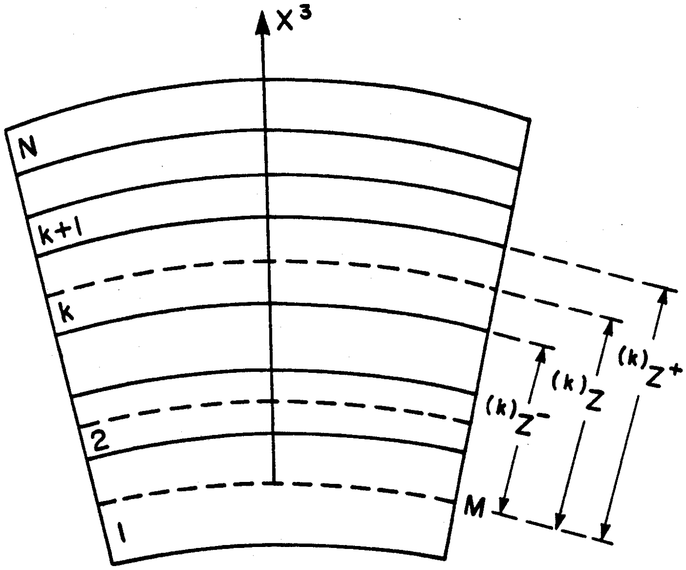


Fig. 1.

system $x^i$ $(i = 1, 2, 3)$, where $x^\alpha$ $(\alpha = 1, 2)$ is the set of curvilinear coordinates on $\sigma$, and $x^3$ is the coordinate normal to $\sigma$.

The distance along $x^3$ between the reference surface and the undeformed mid-surface of the $k$th layer is denoted as ${}^{(k)}Z$ with ${}^{(1)}Z \equiv 0$, while ${}^{(k)}Z^+$ and ${}^{(k)}Z^-$ identify the upper and bottom surfaces of the $k$th layer, respectively (Fig. 1).

We denote by $\tau$ the volume of the shell space in the undeformed configuration; $S^+$ (corresponding to $x^3 = {}^{(N)}Z^+$) and $S^-$ (corresponding to $x^3 = {}^{(1)}Z^-$) denote, respectively, the upper and bottom external surfaces of the shell, while $\mathscr{B}$ is the lateral boundary surface of $\tau$ generated by the normals to $\sigma$ along its boundary curve $\mathscr{C}$ (with arc length $s$).

By $\mathscr{B}_f, \mathscr{B}_v, (\mathscr{B} = \mathscr{B}_f \cup \mathscr{B}_v)$ and $\mathscr{C}_f, \mathscr{C}_v, (\mathscr{C} = \mathscr{C}_f \cup \mathscr{C}_v)$, we denote the two parts of $\mathscr{B}$ and $\mathscr{C}$, where stresses and displacements, respectively, are prescribed.

The components of the spatial metric tensor $g_{ij}$ of the undeformed shell space are connected with their 2-D counterparts $a_{\alpha\beta}$ by

$$g_{\alpha\beta} = \mu^\lambda_\alpha \mu^\omega_\beta a_{\lambda\omega}, \qquad g_{\alpha 3} = g^{\alpha 3} = 0, \qquad g_{33} = g^{33} = 1, \tag{1}$$

where $\mu^\alpha_\beta$ denotes the shifter tensor defined by

$$\mu^\alpha_\beta = \delta^\alpha_\beta - x^3 b^\alpha_\beta, \tag{2}$$

$\delta^\alpha_\beta$ being the Kronecker delta and $b_{\alpha\beta}$ is the second fundamental form of the reference surface. As shown in Eq. (26), $\mu^\alpha_\beta$ is non-singular. Its inverse, denoted by $(\mu^{-1})^\alpha_\beta$, and satisfying the orthogonality relation

$$\mu^\alpha_\beta (\mu^{-1})^\beta_\lambda = \delta^\alpha_\lambda, \tag{3}$$

may be expressed as

$$(\mu^{-1})^\alpha_\beta = \sum_{n=0}^{\infty} (x^3)^n (b^n)^\alpha_\beta. \tag{4}$$

In Eq. (4), $(b^n)^\alpha_\beta$ is defined by

$$(b^n)^\alpha_\beta = b^\lambda_\beta (b^{n-1})^\alpha_\lambda = b^\alpha_\lambda (b^{n-1})^\lambda_\beta, \tag{5}$$

where, in addition,

$$(b^n)^\alpha_\beta = \begin{cases} \delta^\alpha_\beta & \text{for } n = 0, \\ 0 & \text{for } n < 0. \end{cases} \tag{6}$$

As was emphasized by Naghdi (1963) and Librescu (1975a,b), $\mu^\alpha_\beta$ and $(\mu^{-1})^\alpha_\beta$ (called shifters and its inverse, respectively) play an important role in establishing the relationships between space tensor components and their surface (shifted) counterparts.

In order to reduce the 3-D elasticity problem to an equivalent 2-D one, the equations connecting the covariant derivatives of space tensors with their surface counterparts are used. Several of these relationships are displayed next

$$T_{\alpha||\beta} = \mu^\mu_\alpha(\overline{T}_{\mu|\beta} - b_{\mu\beta}\overline{T}_3), \qquad T_{\alpha||3} = \mu^\mu_\alpha \overline{T}_{\mu,3}, \qquad T_{3||\alpha} = \overline{T}_{3,\alpha} + b^\epsilon_\alpha \overline{T}_\epsilon, \qquad T_{3||3} = \overline{T}_{3,3}, \tag{7}$$

where the shifted components are identified by a superposed bar. These can be functions also of the transversal coordinate $x^3$ also. A full account of the relationships between the derivatives of space and surface tensors was provided in the monograph by Librescu (1975a,b).

For shallow shell theory, we may appropriately consider

$$\mu^\alpha_\beta \to \delta^\alpha_\beta, \tag{8}$$

and as a result,

$$\mu \equiv |\mu^\alpha_\beta| = (g/a)^{1/2} = 1 - 2Hx^3 + K(x^3)^2 \to 1, \tag{9}$$

where $g \equiv \det(g_{ij})$ and $a \equiv \det(a_{\alpha\beta})$, while $H$ and $K$ denote the average and Gaussian curvatures of $\sigma$, respectively. By virtue of Eq. (8), it follows that for *shallow shells*

$$g_{ij} \equiv a_{ij}, \qquad g^{ij} = a^{ij}, \tag{10}$$

and as a result, the metric tensors associated with the system of coordinates on $\sigma$ and with its projection on the plane $P$ are the same and, in addition, the curvature tensor of the reference surface behaves as a constant in the differentiation operation.

At the boundary curve $\mathscr{C}$ of $\sigma$, we define the unit tangent and outward normal vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$, respectively, by

$$\boldsymbol{\tau} = \tau^\alpha \mathbf{a}_\alpha, \qquad \boldsymbol{\nu} = \nu^\alpha \mathbf{a}_\alpha = \boldsymbol{\tau} \times \mathbf{n}. \tag{11}$$

Here and in the following, partial differentiation is denoted by a comma $(\cdot)_{,i} \equiv \partial(\cdot)/\partial x_i$, while $(\ )||_i$ and $(\ )|_\alpha$ stand for the covariant differentiations with respect to the space and surface metric, respectively. In the above relationships (as well as in the forthcoming ones), the usual summation convention for repeated indices is implied, where Latin indices range from 1 to 3, while Greek indices range from 1 to 2. In addition, superscript $k$ in brackets attached to any quantity identifies its affiliation with the $k$th layer.

## 3. Kinematic equations

Within the concept of infinitesimal small strains, the 3-D strain–displacement relationships assume the form

$$2E^{(k)}_{ij} = V^{(k)}_{i||j} + V^{(k)}_{j||i}, \tag{12}$$

where $V_i^{(k)}(\equiv V_i^{(k)}(x^\omega, x^3; t))$ denote the 3-D displacement components of the points of the $k$th layer.

Eq. (12) in conjunction with Eq. (7) yields the 3-D strain components under the form

$$\begin{aligned} 2E_{\alpha\beta}^{(k)} &= \mu_\alpha^\lambda\left(\overline{V}_{\lambda|\beta}^{(k)} - b_{\lambda\beta}\overline{V}_3^{(k)}\right) + \mu_\beta^\lambda\left(\overline{V}_{\lambda|\alpha}^{(k)} - b_{\lambda\alpha}\overline{V}_3^{(k)}\right), \\ 2E_{\alpha 3}^{(k)} &= \mu_\alpha^\lambda\overline{V}_{\lambda|3}^{(k)} + \overline{V}_{3,\alpha}^{(k)} + b_\alpha^\lambda\overline{V}_\lambda^{(k)}, \\ E_{33}^{(k)} &= \overline{V}_{3|3}^{(k)}. \end{aligned} \tag{13a–c}$$

In these equations, $\overline{V}_\alpha$ and $\overline{V}_3$ are the shifted displacement components, their spatial counterparts being $V_\alpha$ and $V_3$, respectively. The relationships between these components are (Naghdi, 1963 and Librescu, 1975a,b)

$$V_\alpha = \mu_\alpha^\beta\overline{V}_\beta, \qquad V_3 = \overline{V}_3, \tag{14}$$

where $\overline{V}_\alpha = \overline{V}_\alpha(x^\alpha, x^3)$ and $\overline{V}_3 = \overline{V}_3(x^\alpha, x^3)$.

As concerns the imperfect bonding conditions, these are expressed in the simplest way, by postulating that the jump in normal and tangential displacements is proportional to the associated traction components, or in an equivalent form as

$$\begin{aligned} \hat{\sigma}^{\alpha 3} &= \Gamma_\lambda^\alpha[\![V^\lambda]\!], \\ \hat{\sigma}^{33} &= \Gamma_3^3[\![V^3]\!]. \end{aligned} \tag{15a, b}$$

Herein, $[\![\cdot]\!]$ denotes the jump operator of the quantity "$\cdot$" across the contiguous damaged interfaces. As a result

$$\begin{aligned} [\![V^\lambda]\!] &= V^\lambda(0^-) - V^\lambda(0^+) \equiv \hat{V}^\lambda(x^\omega), \\ [\![V^3]\!] &= V^3(0^-) - V^3(0^+) \equiv \hat{V}^3(x^\omega) \end{aligned} \tag{16a, b}$$

denote, respectively, the jump of tangential and transversal displacements across the interfaces ${}^{(k)}Z^+$ and ${}^{(k+1)}Z^-$; $\Gamma_\lambda^\alpha$ and $\Gamma_3^3$ are the bonding stiffness tensors of the interface between the $k$th and $(k+1)$th layers (with dimensions stress/length), where $0^+ \equiv (x^\alpha, x^3 = {}^{(k)}Z^+)$ and $0^- \equiv (x^\alpha, x^3 = {}^{(k+1)}Z^-)$ define the interfaces weakened by the flaw, whereas $\hat{\sigma}^{\alpha 3}$ and $\hat{\sigma}^{33}$ denote the shear and normal tractions at the respective interfaces.

Eq. (15a,b) reveal that for $\Gamma_\lambda^\alpha = \infty$ and $\Gamma_3^3 = \infty$, the displacement jumps vanish, implying perfectly bonded interfaces. At the other extremity, $\Gamma_\lambda^\alpha = 0$ and $\Gamma_3^3 = 0$ correspond to zero interface tractions, implying completely debonded interfaces. Any finite positive value of interface stiffness parameters defines an imperfect interface. For the special case (Benveniste, 1984) $\Gamma_\lambda^\alpha = 0$, and $\Gamma_3^3 = \infty$, implying that the transversal displacement and the normal traction remain continuous although the shear tractions at the respective interface vanish, complete slip without friction in the tangential directions is allowed. For this case, the terminology of *perfectly lubricated interfaces* was afforded (Aboudi, 1987).

In the following developments, instead of the previously bonding stiffness quantities, the imperfect bonding is described in terms of the compliance bonding tensors $B_\lambda^\alpha$ and $B_3^3$ (of dimensions length/stress). These are the inverse of the bonding stiffness tensors $\Gamma_\lambda^\alpha$ and $\Gamma_3^3$, respectively, in the sense of (Schoenberg, 1980)

$$\begin{aligned} (\Gamma^{-1})_\lambda^\alpha B_\alpha^\omega &= \delta_\lambda^\omega, \\ (\Gamma^{-1})_3^3 B_3^3 &= 1. \end{aligned} \tag{17a, b}$$

In this case, the inverted form of Eq. (15a,b) becomes

$$[\![V^{\alpha}]\!] = B^{\alpha}_{\lambda}\hat{\sigma}^{\lambda 3}, \tag{18a}$$

$$[\![V^{3}]\!] = B^{3}_{3}\hat{\sigma}^{3}_{3}. \tag{18b}$$

In order to model the theory of laminated shells featuring such interlaminae imperfections, and at the same time to obtain the simplest possible system of governing equations, an extension of the representation of shifted displacement components advanced in various contexts and with various degrees of approximation, starting with that considered in the earlier work by Liu et al. (1994), and continuing with those by Schmidt and Librescu (1996), Cheng et al. (1996a,b, 1997, 2000), Di Sciuva (1997), Williams and Addessio (1997, 1998), Di Sciuva et al. (1999), Cheng and Kitipornchai (1998), Librescu et al. (1999), and Icardi et al. (2000) is used as

$$\overline{V}^{(k)}_{\alpha}(x^{\omega}x^{3};t) = v_{\alpha} + x^{3}\psi_{\alpha} + \sum_{l=1}^{k-1}\left[x^{3} - {}^{(l)}Z^{+}\right]\Omega^{(l)}_{\alpha}\ Y\left(x^{3} - {}^{(l)}Z^{+}\right) + \sum_{l=1}^{k-1}{}^{(l)}\hat{v}_{\alpha}Y\left(x^{3} - {}^{(l)}Z^{+}\right), \tag{19a}$$

$$\overline{V}^{(k)}_{3}(x^{\omega}x^{3};t) = v_{3} + x^{3}\psi_{3} + \sum_{l=1}^{k-1}\left[x^{3} - {}^{(l)}Z^{+}\right]\Omega^{(l)}_{3}\ Y\left(x^{3} - {}^{(l)}Z^{+}\right) + \sum_{l=1}^{k-1}{}^{(l)}\hat{v}_{3}Y\left(x^{3} - {}^{(l)}Z^{+}\right). \tag{19b}$$

Herein, ${}^{(\ell)}\hat{v}_{\alpha}$ and ${}^{(\ell)}\hat{v}_{3}$ are the shifted counterparts of the tangential and normal displacement jumps across the top surface of the $\ell$th layer, while $Y(x^{3} - {}^{(\ell)}Z^{+}) \equiv Y_{\ell}$ denotes Heaviside's step distribution. In addition, $v_{\alpha}(\equiv v_{\alpha}(x^{\omega};t))$ and $v_{3}(\equiv v_{3}(x^{\omega};t))$ denote the displacement components of the reference surface of the shell (defined by $x^{3} = 0$), $\psi_{\alpha}(\equiv \psi_{\alpha}(x^{\omega},t))$ denote the rotations of the normal to the reference surface, $\psi_{3}(\equiv \psi_{3}(x^{\omega},t))$ corresponds to the deformation in the transverse normal direction, while $\Omega^{(\ell)}_{\alpha}(\equiv \Omega^{(\ell)}_{\alpha}(x^{\omega};t))$, and $\Omega^{(\ell)}_{3}(\equiv \Omega^{(\ell)}_{3}(x^{\omega},t))$ are functions that are determined by fulfilling the continuity conditions for shear and normal tractions across laminae interfaces.

Whereas the first two terms in Eq. (19a) and the first term in Eq. (19b) represent the standard contributions in the displacement representations used in the modeling of the first-order transverse shear deformation theory of laminated shells, the third term in the same equation supplies the contributions, which are continuous with respect to the $x^{3}$ coordinate, but with jumps in the first derivatives at the interfaces between the consecutive layers. This representation constitutes an extension of that prompted by Di Sciuva (1987), and of that advanced by Ossadzow et al. (1995), in the case of laminated composite shells featuring perfectly bonded interfaces. Moreover, the second term in Eq. (19b) captures the effect of transverse normal compressibility, which for the present case is very important.

It should be remarked that in the absence of displacement jumps, the displacement components $\overline{V}^{(k)}_{\alpha}$ and $\overline{V}^{(k)}_{3}$ become continuous functions of $x^{3}$ for arbitrary values of $\Omega^{(\ell)}_{\alpha}$ and $\Omega^{(\ell)}_{3}$.

By virtue of Eqs. (13a–c), and in conjunction with Eqs. (19a) and (19b), the 3-D tangential strain components can be expressed as

$$E^{(k)}_{\alpha\beta} = \overset{0}{e}{}^{(k)}_{\alpha\beta} + x^{3}\ \overset{1}{e}{}^{(k)}_{\alpha\beta} + (x^{3})^{2}\ \overset{2}{e}{}^{(k)}_{\alpha\beta}, \tag{20}$$

where $\overset{m}{e}_{\alpha\beta} = \overset{m}{e}_{\alpha\beta}(x^{\omega};t)$ are the tangential 2-D strain measures. Their expressions are

$$2\overset{0}{e}{}^{(k)}_{\alpha\beta} = v_{\alpha|\beta} + v_{\beta|\alpha} - 2b_{\alpha\beta}v_{3} - \underset{\text{- - - - - - - - - - - - - - - -}}{\sum_{\ell=1}^{k-1}{}^{(\ell)}Z^{+}\left(\Omega^{(\ell)}_{\alpha|\beta} + \Omega^{(\ell)}_{\beta|\alpha}\right)} + \underset{\text{- - - - - - - - - - -}}{2b_{\alpha\beta}\sum_{\ell=1}^{k-1}{}^{(\ell)}Z^{+}\Omega^{(\ell)}_{3}} + \underline{\sum_{\ell=1}^{k-1}\left({}^{(\ell)}\hat{v}_{\alpha|\beta} + {}^{(\ell)}\hat{v}_{\beta|\alpha}\right)}$$
$$\underline{- 2b_{\alpha\beta}\sum_{\ell=1}^{k-1}{}^{(\ell)}\hat{v}_{3}}, \tag{21a}$$

$$2\overset{1}{e}{}^{(k)}_{\alpha\beta} = \psi_{\alpha|\beta} + \psi_{\beta|\alpha} - 2b_{\alpha\beta}\psi_3 - b^{\gamma}_{\alpha}v_{\gamma|\beta} - b^{\gamma}_{\beta}v_{\gamma|\beta} + 2b^{\gamma}_{\alpha}b_{\gamma\beta}v_3 + \sum_{\ell=1}^{k-1}\left(\Omega^{(\ell)}_{\beta|\alpha} + \Omega^{(\ell)}_{\alpha|\beta}\right)$$
$$+ \; b^{\gamma}_{\alpha}\sum_{\ell=1}^{k-1}{}^{(\ell)}Z^{+}\Omega^{(\ell)}_{\gamma|\beta} + b^{\gamma}_{\beta}\sum_{\ell=1}^{k-1}{}^{(\ell)}Z^{+}\Omega^{(\ell)}_{\gamma|\alpha} - 2b_{\alpha\beta}\sum_{\ell=1}^{k-1}\Omega^{(\ell)}_{3} \; - \; 2b^{\gamma}_{\alpha}b_{\gamma\beta}\sum_{\ell=1}^{k-1}{}^{(\ell)}Z^{+}\Omega^{(\ell)}_{3}$$
$$- b^{\gamma}_{\alpha}\sum_{\ell=1}^{k-1}{}^{(\ell)}\hat{v}_{\gamma|\beta} - b^{\gamma}_{\beta}\sum_{\ell=1}^{k-1}{}^{(\ell)}\hat{v}_{\gamma|\alpha} + 2b^{\gamma}_{\alpha}b_{\gamma\beta}\sum_{\ell=1}^{k-1}{}^{(\ell)}\hat{v}_{3}, \tag{21b}$$

$$2\overset{0}{e}{}^{(k)}_{\alpha\beta} = -\left\{ b^{\gamma}_{\alpha}\left(\psi_{\gamma|\beta} + \sum_{\ell=1}^{k-1}\Omega^{(\ell)}_{\gamma|\beta}\right) + b^{\gamma}_{\beta}\left(\psi_{\gamma|\alpha} + \sum_{\ell=1}^{k-1}\Omega^{(\ell)}_{\gamma|\alpha}\right) - 2b^{\gamma}_{\alpha}b_{\gamma\beta}\left(\psi_3 + \sum_{\ell=1}^{k-1}\Omega^{(\ell)}_{3}\right)\right\}. \tag{21c}$$

As concerns the transverse shear strains ${}^{(k)}E_{\alpha 3}$, their expression is

$$E^{(k)}_{\alpha 3} = \overset{0}{e}{}^{(k)}_{\alpha 3} + x^3\overset{1}{e}{}^{(k)}_{\alpha 3}, \tag{22}$$

where

$$2\overset{0}{e}{}^{(k)}_{\alpha 3}(\equiv \gamma^{(k)}_{\alpha 3}) = \psi_{\alpha} + v_{3,\alpha} + b^{\lambda}_{\alpha}v_{\lambda} + \sum_{\ell=1}^{k-1}{}^{(\ell)}\mu^{\gamma}_{\alpha}\Omega^{(\ell)}_{\gamma} \; - \; \sum_{\ell=1}^{k-1}{}^{(\ell)}Z^{+}\Omega^{(\ell)}_{3|\alpha} \; + b^{\gamma}_{\alpha}\sum_{\ell=1}^{k-1}{}^{(\ell)}\hat{v}_{\gamma} + \sum_{\ell=1}^{k-1}{}^{(\ell)}\hat{v}_{3|\alpha}, \tag{23a}$$

and

$$2\overset{1}{e}{}^{(k)}_{\alpha 3} = \psi_{3|\alpha} + \sum_{\ell=1}^{k-1}\Omega^{(\ell)}_{3|\alpha}. \tag{23b}$$

Finally, the transverse normal strain component is expressed as

$$E^{(k)}_{33} \equiv \overset{0}{e}{}^{(k)}_{33} = \psi_3 + \sum_{\ell=1}^{k-1}\Omega^{(\ell)}_{3}. \tag{24}$$

In the previously displayed equations $\overset{0}{e}_{\alpha\beta}$, $(\overset{1}{e}_{\alpha\beta}, \overset{2}{e}_{\alpha\beta})$, $(\overset{0}{e}_{\alpha 3}, \overset{1}{e}_{\alpha 3})$ and $\overset{0}{e}_{33}$ define the membrane, bending, transverse shear, and transverse normal strain components, respectively. In addition,

$${}^{(\ell)}\mu^{\gamma}_{\alpha} \equiv \delta^{\gamma}_{\alpha} - {}^{(\ell)}Z^{+}b^{\gamma}_{\alpha} \tag{25a}$$

denotes the shifter tensor generalized to laminated composite shells (Librescu and Schmidt, 1991).

At this point, it is worth noting that both the representation of the displacement field, Eqs. (19a) and (19b), and the expressions of 2-D strain measures $\overset{r}{e}_{ij}$, can be viewed as the superposition of three different contributions, namely, (i) of one that is similar to that characterizing the *extended* first-order shear deformation theory in the sense of including the effect of thickness compressibility, (ii) of the one exhibiting a continuous, piecewise variation from layer to layer being related to the functions $\Omega^{(\ell)}_{\alpha}$ and $\Omega^{(\ell)}_{3}$, contributions underscored by a broken line, and finally, (iii) that part related to the effect of imperfectly bonded interfaces, which involves the displacement jumps ${}^{(k)}\hat{v}_{\alpha}$ and ${}^{(k)}\hat{v}_{3}$, marked by a solid line.

Eqs. (21a)–(21c) supply the membrane and bending 2-D *symmetric* strain measures, $\overset{0}{e}{}^{(k)}_{\alpha\beta}$, $\overset{1}{e}{}^{(k)}_{\alpha\beta}$, and $\overset{2}{e}{}^{(k)}_{\alpha\beta}$.

However, alternative, *non-symmetric* strain measure counterparts can be defined as

$$\gamma_{\alpha\beta}^{(k)} = v_{\alpha|\beta} - b_{\alpha\beta}v_3 - \underline{\sum_{\ell=1}^{k-1} {}^{(\ell)}Z^{+}\Omega_{\alpha|\beta}^{(\ell)}} + \underline{b_{\alpha\beta}\sum_{\ell=1}^{k-1} {}^{(\ell)}Z^{+}\Omega_{3}^{(\ell)}} + \underline{\sum_{\ell=1}^{k-1} {}^{(\ell)}\hat{v}_{\alpha|\beta}} - \underline{b_{\alpha\beta}\sum_{\ell=1}^{k-1} {}^{(\ell)}\hat{v}_{3}}, \tag{25b}$$

$$\kappa_{\alpha\beta}^{(k)} = \psi_{\alpha|\beta} - b_{\alpha\beta}\psi_3 + \underline{\sum_{\ell=1}^{k-1} \Omega_{\alpha|\beta}^{(\ell)}} - \underline{b_{\alpha\beta}\sum_{\ell=1}^{k-1} \Omega_{3}^{(\ell)}} \tag{25c}$$

by virtue of which

$$2E_{\alpha\beta}^{(k)} = \mu_{\alpha}^{\gamma}\left(\gamma_{\gamma\beta}^{(k)} + x^3\kappa_{\gamma\beta}^{(k)}\right) + \mu_{\beta}^{\gamma}\left(\gamma_{\gamma\alpha}^{(k)} + x^3\kappa_{\gamma\alpha}^{(k)}\right). \tag{26}$$

It can readily be shown that between the previously defined symmetric strain measures and the non-symmetric ones, $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$, the following relationships can be established (Librescu, 1975a,b):

$$\overset{0}{e}{}_{\alpha\beta}^{(k)} = \tfrac{1}{2}\left(\gamma_{\alpha\beta}^{(k)} + \gamma_{\beta\alpha}^{(k)}\right), \tag{27a}$$

$$\overset{1}{e}{}_{\alpha\beta}^{(k)} = \tfrac{1}{2}\left(\kappa_{\alpha\beta}^{(k)} + \kappa_{\beta\alpha}^{(k)} - \underbrace{b_{\alpha}^{\rho}\gamma_{\rho\beta}^{(k)} - b_{\beta}^{\rho}\gamma_{\rho\alpha}^{(k)}}\right), \tag{27b}$$

$$\overset{2}{e}{}_{\alpha\beta}^{(k)} = -\tfrac{1}{2}\left(\underbrace{b_{\alpha}^{\rho}\,\kappa_{\rho\beta}^{(k)} + b_{\beta}^{\rho}\,\kappa_{\rho\alpha}^{(k)}}\right). \tag{27c}$$

In addition, it is easily seen that $\overset{2}{e}{}_{\alpha\beta}^{(k)}$ is not independent, in the sense that it can be expressed in terms of $\overset{0}{e}{}_{\alpha\beta}^{(k)}$ and $\overset{1}{e}{}_{\alpha\beta}^{(k)}$ as

$$\overset{2}{e}{}_{\alpha\beta}^{(k)} = -b_{\alpha}^{\rho}b_{\beta}^{\sigma}\,\overset{0}{e}{}_{\rho\sigma}^{(k)} - \tfrac{1}{2}\left(b_{\beta}^{\rho}\overset{1}{e}{}_{\rho\alpha}^{(k)} + b_{\alpha}^{\rho}\,\overset{1}{e}{}_{\rho\beta}^{(k)}\right). \tag{28}$$

## 4. Special cases

### *4.1. Strain–displacement relationships for shallow-shell theory*

For shallow shells, by virtue of Eq. (8) and of

$${}^{(\ell)}\mu_{\beta}^{\alpha} \to \delta_{\beta}^{\alpha}, \qquad {}^{(\ell)}(\mu^{-1})_{\beta}^{\alpha} \to \delta_{\beta}^{\alpha}, \tag{29}$$

a simplification of Eqs. (21a)–(21c) is obtained:

$$2\overset{0}{e}{}_{\alpha\beta}^{(k)} = v_{\alpha|\beta} + v_{\beta|\alpha} - 2b_{\alpha\beta}v_3 - \sum_{\ell=1}^{k-1} {}^{(\ell)}Z^{+}\underline{\left(\Omega_{\alpha|\beta}^{(\ell)} + \Omega_{\beta|\alpha}^{(\ell)}\right)} + 2b_{\alpha\beta}\sum_{\ell=1}^{k-1} \underline{{}^{(\ell)}Z^{+}\Omega_{3}^{(\ell)}} + \sum_{\ell=1}^{k-1} \underline{\left({}^{(\ell)}\hat{v}_{\alpha|\beta} + {}^{(\ell)}\,\hat{v}_{\beta|\alpha}\right)} - 2b_{\alpha\beta}\sum_{\ell=1}^{k-1} \underline{{}^{(\ell)}\hat{v}_{3}}, \tag{30a}$$

$$2\overset{1}{e}{}_{\alpha\beta}^{(k)} = \psi_{\alpha|\beta} + \psi_{\beta|\alpha} - 2b_{\alpha\beta}\psi_3 + \underline{\sum_{\ell=1}^{k-1}\left(\Omega_{\beta|\alpha}^{(\ell)} + \Omega_{\alpha|\beta}^{(\ell)}\right)} - 2b_{\alpha\beta}\underline{\sum_{\ell=1}^{k-1}\Omega_{3}^{(\ell)}}, \tag{30b}$$

$$2\overset{2}{e}{}^{(k)}_{\alpha\beta} = 0, \tag{30c}$$

$$2\overset{0}{e}{}^{(k)}_{\alpha 3} (\equiv \gamma_{\alpha 3}) = \psi_\alpha + v_{3,\alpha} + \underset{- - - - -}{\sum_{\ell=1}^{k-1} \Omega^{(\ell)}_\alpha} - \underset{- - - - - - - - - -}{\sum_{\ell=1}^{k-1} (\ell) Z^+ \Omega^{(\ell)}_{3|\alpha}} + \underline{b^\gamma_\alpha \sum_{\ell=1}^{k-1} (\ell) \hat{v}_\gamma} + \underline{\sum_{\ell=1}^{k-1} (\ell) \hat{v}_{3|\alpha}}, \tag{31a}$$

$$2\overset{1}{e}{}^{(k)}_{\alpha 3} = \psi_{3|\alpha} + \sum_{\ell=1}^{k-1} \Omega^{(\ell)}_{3|\alpha}. \tag{31b}$$

It is also seen that, by virtue of Eq. (8), Eq. (26) becomes

$$2E^{(k)}_{\alpha\beta} = \left(\gamma^{(k)}_{\alpha\beta} + \gamma^{(k)}_{\beta\alpha}\right) + x^3 \left(\kappa^{(k)}_{\alpha\beta} + \kappa^{(k)}_{\beta\alpha}\right), \tag{32}$$

wherefrom it clearly appears that in this case the terms in Eqs. (27a)–(27c) underlined by an undulated line should be discarded.

As concerns the transverse normal strain measure, its expression given by Eq. (24) remains unchanged.

## 5. Determination of the functions $\overset{(\ell)}{_3}$ and $\overset{(\ell)}{_\alpha}$

As previously mentioned, the terms involving $\Omega^{(\ell)}_\alpha$ and $\Omega^{(\ell)}_3$ in Eqs. (19a)–(19b) have to be determined from the requirements of shear and normal traction continuity across the layer interfaces.

As a preparatory step, the relationships between the transverse shear and transverse normal stresses, respectively, and the corresponding strain components are considered. As a result

$${}^{(k)}\sigma^{\beta 3} = 2\,{}^{(k)}E^{\beta 3\alpha 3} E^{(k)}_{\alpha 3}, \tag{33a}$$

$${}^{(k)}\sigma^{33} = {}^{(k)}E^{3333} E^{(k)}_{33} + {}^{(k)} \underbrace{E^{33\alpha\beta} E^{(k)}_{\alpha\beta}} \tag{33b}$$

are used in conjunction with Eqs. (21a)–(21c), (23a) and (23b) and (24). However, keeping in view that in the expressions of $\overset{r}{e}{}^{(k)}_{\alpha\beta}$, the functions $\Omega^{(k)}_\alpha$ appear in differential form, whereas $\Omega^{(k)}_3$ appear only in an algebraic form, for the purpose of determination of $\Omega^{(k)}_3$ from Eq. (33b), we will use a fact that is particularly true in the case of composite materials, namely that the effect of $E_{\alpha\beta}$ on $\sigma^{33}$ is much smaller than that of $E_{33}$. As a result, *in the context of determining* $\Omega^{(k)}_3$ *only*, the term identified in Eq. (33b) by the undulated line will be neglected. In such a way, the functions $\Omega^{(k)}_3$ and $\Omega^{(k)}_\alpha$ are determined separately from Eqs. (33a) and (33b), respectively as

$$\Omega^{(k)}_3 = {}^{(1)}E^{3333} \left({}^{(k+1)}F_{3333} - {}^{(k)}F_{3333}\right) \psi_3, \tag{34a}$$

$$\begin{aligned} \Omega^{(k)}_\alpha = {}^{(k)}(\mu^{-1})^\rho_\alpha \Bigg\{ & 4\,{}^{(1)}E^{\beta 3\omega 3} \left[{}^{(k+1)}F_{\beta 3\rho 3} - {}^{(k)}F_{\beta 3\rho 3}\right] \left(\psi_\omega + v_{3|\omega} + b^\sigma_\omega v_\sigma + {}^{(1)}Z^+ \psi_{3|\omega}\right) \\ & + 4 \sum_{\ell=2}^{k} {}^{(\ell)}E^{\beta 3\omega 3} \left({}^{(k+1)}F_{\beta 3\rho 3} - {}^{(k)}F_{\beta 3\rho 3}\right) \left({}^{(\ell)}Z^+ - {}^{(\ell-1)}Z^+\right) \left(\psi_{3|\omega} + \underset{- - -}{\sum_{n=1}^{\ell-1} \Omega^{(n)}_{3|\omega}}\right) \\ & - \underline{{}^{(k)}\hat{v}_{3|\rho}} - \underline{b^\lambda_\rho {}^{(k)}\hat{v}_\lambda} \Bigg\}, \end{aligned} \tag{34b}$$

In Eqs. (33a) and (33b) and (34a) and (34b), $E^{\alpha 3\omega 3}, E^{3333}$, and $F_{\beta 3\alpha 3}$ are the components in transverse shear and transverse normal directions of elastic and compliance tensors, $E^{ijmn}$ and $F_{ijmn}$, respectively.

A more condensed form of Eqs. (34a) and (34b) is obtained when the following notations involving the elastic characteristics, are considered:

$$^{(\ell;k)}\mathscr{P}^{\omega}_{\cdot\alpha} = {}^{(k)}(\mu^{-1})^{\sigma}_{\alpha}\,{}^{(\ell;k)}\Lambda^{\omega}_{\cdot\alpha}, \tag{35a}$$

where

$$^{(\ell;k)}\Lambda^{\omega}_{\cdot\sigma} = 4^{(\ell)}E^{\beta3\omega3}\left({}^{(k+1)}F_{\beta3\sigma3} - {}^{(k)}F_{\beta3\sigma3}\right), \tag{35b}$$

and

$$^{(k)}\Lambda^{3}_{\cdot3} = {}^{(1)}E^{3333}\left({}^{(k+1)}F_{3333} - {}^{(k)}F_{3333}\right). \tag{36}$$

Using Eqs. (35a) and (35b) and (36), Eqs. (34a) and (34b) become

$$\Omega^{(k)}_{3} = {}^{(k)}\Lambda^{3}_{\cdot3}\psi_{3}, \tag{37a}$$

$$\Omega^{(k)}_{\alpha} = \left(v_{3|\omega} + b^{\lambda}_{\omega}v_{\lambda} + \psi_{\omega} + {}^{(1)}Z^{+}\psi_{3|\omega}\right){}^{(1;k)}\mathscr{P}^{\omega}_{\cdot\alpha} + \sum_{\ell=2}^{k}\left({}^{(\ell)}Z^{+} - {}^{(\ell-1)}Z^{+}\right)\left(1 + \sum_{n=1}^{\ell-1}{}^{(n)}\Lambda^{3}_{\cdot3}\right){}^{(\ell;k)}\mathscr{P}^{\omega}_{\cdot\alpha}\psi_{3|\omega} - {}^{(k)}\left(\mu^{-1}\right)^{\omega}_{\alpha}\left(b^{\lambda}_{\omega}\hat{v}^{(k)}_{\lambda} + \hat{v}^{(k)}_{3|\omega}\right). \tag{37b}$$

Within the general shell theory, these expressions reveal that the functions $\Omega^{(k)}_{\alpha}$ are expressed in terms of the unknown functions $\psi_{\alpha}$, $\psi_{3}$, $v_{\alpha}$ and $v_{3}$, as well as of the displacement jumps $\hat{v}_{\alpha}$ and $\hat{v}_{3}$, whereas $\Omega^{(k)}_{3}$ is expressed solely in terms of the function $\psi_{3}$. Herein, ${}^{(k)}(\mu^{-1})^{\rho}_{\alpha} \equiv (\mu^{-1})^{\rho}_{\alpha}(x^{\alpha}, x^{3} = {}^{(k)}Z^{+})$ is the inverse shifter for laminated shells whose expression (Librescu and Schmidt, 1991), is

$$^{(k)}(\mu^{-1})^{\rho}_{\alpha} = \frac{\delta^{\rho}_{\alpha} + {}^{(k)}Z^{+}(b^{\rho}_{\alpha} - 2H\delta^{\rho}_{\alpha})}{1 - 2({}^{(k)}Z^{+})H + ({}^{(k)}Z^{+})^{2}K}. \tag{38}$$

The denominator in Eq. (38) can be expressed in a more compact form as

$$1 - 2({}^{(k)}Z^{+})H + ({}^{(k)}Z^{+})^{2}K = \det({}^{(k)}\mu^{\alpha}_{\beta}) \equiv {}^{(k)}\mu, \tag{39}$$

where ${}^{(k)}\mu^{\alpha}_{\beta}$ is defined by Eq. (25a).

It can also be remarked that for *shallow* shells and flat plates, $\Omega^{(k)}_{3}$ remains unchanged, whereas Eq. (37b) reduces to

$$\Omega^{(k)}_{\alpha} = {}^{(1;k)}\Lambda^{\omega}_{\cdot\alpha}(\psi_{\omega} + v_{3,\omega} + {}^{(1)}Z^{+}\psi_{3|\omega}) + \sum_{\ell=2}^{k}\left({}^{(\ell)}Z^{+} - {}^{(\ell-1)}Z^{+}\right)\left(1 + \sum_{n=1}^{\ell-1}{}^{(n)}\Lambda^{3}_{\cdot3}\right){}^{(\ell;k)}\Lambda^{\omega}_{\cdot\alpha}\psi_{3|\omega} - \hat{v}^{(k)}_{3|\alpha} \tag{40}$$

Eq. (40) shows that in this special case, the expression of $\Omega^{(k)}_{\alpha}$ does no longer include the displacement jump $\hat{v}_{\alpha}$. The same equation also reveals that in the case of perfectly bonded interfaces, and when transverse normal compressibility is disregarded, Eq. (40) coincides with its specialized counterpart derived by Librescu and Schmidt (1991).

## 6. Expression of displacement jumps

In order to represent the governing equations in terms of the displacement quantities, the continuity functions $\Omega^{(k)}_{\alpha}$, $\Omega^{(k)}_{3}$, and displacement jumps ${}^{(k)}\hat{v}_{3}$ and ${}^{(k)}\hat{v}_{\lambda}$ have to be expressed entirely in terms of $\psi_{\alpha}$, $\psi_{3}$, $v_{\alpha}$ and $v_{3}$. Concerning $\Omega^{(k)}_{3}$, this step was materialized in Eq. (37a). Preparatory to expressing the other quantities in terms of displacements and rotations only, determination of transverse shear and transversal normal strain components, and implicitly of shear and normal tractions is required. Using Eq. (33a) in conjunction with Eqs. (23a) and (18a), after lengthy but straightforward manipulations, one obtains

$$
{}^{(k)}\hat{V}^{\alpha} = {}^{(k)}B^{\alpha(k)}_{\lambda}E^{\lambda 3\omega 3}\left(v_{3,\rho} + b^{\sigma}_{\rho}v_{\sigma} + \psi_{\rho} + {}^{(k)}Z^{+}\psi_{3|\rho}\right)\left(\delta^{\rho}_{\omega} + \sum_{\ell=1}^{k}{}^{(1;\ell)}\Lambda^{\rho}_{\cdot\omega}\right)
+ \sum_{\ell=1}^{k}\left[\left({}^{(k)}Z^{+} - {}^{(\ell)}Z^{+}\right){}^{(\ell)}\Lambda^{3}_{\cdot 3}\delta^{\rho}_{\omega} + \sum_{m=2}^{\ell}\left({}^{(m)}Z^{+} - {}^{(m-1)}Z^{+}\right)\left(1 + \sum_{n=1}^{m-1}{}^{(n)}\Lambda^{3}_{\cdot 3}\right){}^{(m;\ell)}\Lambda^{\rho}_{\cdot\omega}\right]\psi_{3|\rho}. \tag{41}
$$

It is also important to obtain the expression of the shifted counterparts of ${}^{(k)}\hat{V}^{\alpha}$ given by

$$
{}^{(k)}\hat{v}_{\alpha} = {}^{(k)}(\mu^{-1})^{\rho(k)}_{\alpha}\hat{V}_{\rho} = {}^{(k)}(\mu^{-1})^{\rho(k)}_{\alpha}\hat{g}_{\rho\beta}{}^{(k)}\hat{V}^{\beta}, \tag{42}
$$

where ${}^{(k)}\hat{g}_{\mu\beta} \equiv {}^{(k)}g_{\mu\beta}(x_{\alpha}, x^{3} = {}^{(k)}Z^{+})$. Taking account of Eq. (41) in Eq. (42), the expression of ${}^{(k)}\hat{v}_{\alpha}$ writes

$$
{}^{(k)}\hat{v}_{\alpha} = {}^{(k)}\Psi^{\rho}_{\cdot\alpha}\left(v_{3|\rho} + b^{\sigma}_{\rho}v_{\sigma} + \psi_{\rho}\right) + \left({}^{(k)}Z^{+(k)}\Psi^{\rho}_{\cdot\alpha} + {}^{(k)}\phi^{\rho}_{\cdot\alpha}\right)\psi_{3|\rho}. \tag{43}
$$

Similarly, use of Eq. (24) considered in conjunction with Eq. (33b) in Eq. (18b), yields ${}^{(k)}\hat{v}_{3} \equiv {}^{(k)}\hat{V}_{3}$ as

$$
{}^{(k)}\hat{v}_{3} = {}^{(k)}\Psi^{3}_{\cdot 3}\psi_{3}. \tag{44}
$$

Herein

$$
{}^{(k)}\Psi^{\rho}_{\cdot\alpha} = {}^{(k)}\left(\mu^{-1}\right)^{\sigma}_{\alpha}\ {}^{(k)}\hat{g}_{\sigma\beta}\ {}^{(k)}B^{\beta(k+1)}_{\lambda}E^{\lambda 3\omega 3}\left(\delta^{\rho}_{\omega} + \sum_{\ell=1}^{k}{}^{(1;\ell)}\Lambda^{\rho}_{\cdot\omega}\right), \tag{45a}
$$

$$
{}^{(k)}\phi^{\rho}_{\cdot\alpha} = {}^{(k)}\left(\mu^{-1}\right)^{\mu}_{\alpha}\ {}^{(k)}\hat{g}_{\mu\beta}\ {}^{(k)}B^{\beta}_{\lambda}\ {}^{(k+1)}E^{\lambda 3\omega 3}
\times\sum_{\ell=1}^{k}\left\{\left({}^{(k)}Z^{+} - {}^{(\ell)}Z^{+}\right)\ {}^{(\ell)}\Lambda^{3}_{\cdot 3}\delta^{\rho}_{\omega}\sum_{m=2}^{\ell}\left[\left({}^{(m)}Z^{+} - {}^{(m-1)}Z^{+}\right)\times\left(1 + \sum_{n=1}^{m-1}\ {}^{(n)}\Lambda^{3}_{\cdot 3}\right){}^{(m;\ell)}\Lambda^{\rho}_{\cdot\omega}\right]\right\}, \tag{45b}
$$

$$
{}^{(k)}\Psi^{3}_{\cdot 3} = {}^{(k)}B^{3}_{3}\ {}^{(k+1)}E^{3333}\left(1 + \sum_{\ell=1}^{k}\ {}^{(\ell)}\Lambda^{3}_{\cdot 3}\right). \tag{45c}
$$

In the light of Eqs. (43) and (44), Eq. (40) can be cast as

$$
\Omega^{(k)}_{\alpha} = {}^{(k)}M^{\rho}_{\cdot\alpha}(v_{3|\rho} + b^{\sigma}_{\rho}v_{\sigma} + \psi_{\rho}) + {}^{(k)}N^{\rho}_{\cdot\alpha}\psi_{3|\rho}, \tag{46}
$$

where

$$
{}^{(k)}M^{\rho}_{\cdot\alpha} = {}^{(1;k)}\mathscr{P}^{\rho}_{\cdot\alpha} - {}^{(k)}\mathscr{M}^{\rho}_{\cdot\alpha}, \tag{47a}
$$

$$
{}^{(k)}N^{\rho}_{\cdot\alpha} = {}^{(1)}Z^{+(1;k)}\mathscr{P}^{\rho}_{\cdot\alpha} + \sum_{\ell=2}^{k}\left({}^{(\ell)}Z^{+} - {}^{(\ell-1)}Z^{+}\right)\left(1 + \sum_{n=1}^{\ell-1}\Lambda^{3}_{\cdot 3}\right){}^{(\ell,k)}\mathscr{P}^{\rho}_{\cdot\alpha} - b^{\lambda}_{\omega}{}^{(k)}\left(\mu^{-1}\right)^{\omega}_{\alpha}\left({}^{(k)}Z^{+}\Psi^{\rho}_{\cdot\lambda} + \phi^{\rho}_{\cdot\lambda}\right)
- {}^{(k)}\left(\mu^{-1}\right)^{\rho(k)}_{\alpha}\Psi^{3}_{\cdot 3}, \tag{47b}
$$

$$
{}^{(k)}\mathscr{M}^{\rho}_{\cdot\alpha} = b_{\alpha\sigma}{}^{(k)}B^{\sigma(k)}_{\pi}E^{\pi 3\omega 3}\left(\delta^{\rho}_{\omega} + \sum_{\ell=1}^{k}{}^{(1;\ell)}\Lambda^{\rho}_{\cdot\omega}\right), \tag{47c}
$$

whereas ${}^{(1;\ell)}\mathscr{P}^{\rho}_{\cdot\alpha}$ is expressed by Eq. (35a). For perfectly bonded interfaces implying $B^{\omega}_{\sigma} \to 0$, $B^{3}_{3} \to 0$, and consequently $\left({}^{(k)}\Psi^{\rho}_{\cdot\alpha};\ {}^{(k)}\Psi^{3}_{\cdot 3};\ {}^{(k)}\phi^{\rho}_{\cdot\sigma}\right) \to 0$, it is readily seen that ${}^{(k)}M^{\rho}_{\cdot\alpha} \to {}^{(1;k)}\mathscr{P}^{\rho}_{\cdot\alpha}$, while for shallow shells or flat plates ${}^{(1;k)}\mathscr{P}^{\rho}_{\cdot\alpha} \to {}^{(1;k)}\Lambda^{\rho}_{\cdot\alpha}$.

## 7. The equations of motion and boundary conditions

The Hamilton principle of 3-D elastokinetics will be used to derive the equations of motion and the boundary conditions of laminated shells featuring damaged boundary interfaces. Consistent with this principle

$$\int_{t_1}^{t_2}\left[\int_{\tau}\sigma^{ij}\delta E_{ij}\mathrm{d}\tau-\delta K-\int_{\mathscr{A}}\sigma^{i}\delta V_i\mathrm{d}A-\int_{\tau}\rho_0 H^i\delta V_i\mathrm{d}\tau\right]\mathrm{d}t=0, \tag{48}$$

where $K(\equiv\int_\tau \rho_0\dot{V}^i\ \dot{V}_i\mathrm{d}\tau)$ denotes the kinetic energy, $\mathbf{H}(\equiv H^i\mathbf{g}_i)$ denotes the vector of body forces measured per unit mass of the undeformed body of volume $\tau$, $\rho_0$ denotes the mass density, while $\sigma^i(\equiv\sigma^{ji}n_j)$ are the components of the stress vector prescribed over the undeformed external boundary surface $\mathscr{A}$, where $n_i$ denotes the components of the outward normal unit vector $\mathbf{n}$, and $t_1$ and $t_2$ are two arbitrary instants of time. We make use in Eq. (48) of the subsidary condition

$$2\delta E_{ij}=\delta V_{i\|j}+\delta V_{j\|i}, \tag{49}$$

as well as of Eqs. (19a), (19b) and (8), and of the expression $\mathrm{d}\tau=\mu\,\mathrm{d}x^3\,\mathrm{d}\sigma$. Moreover, keeping in view that the total external area $\mathscr{A}$ is constituted of the lateral bounding area $\mathscr{B}$ (for which $n_\alpha\mathrm{d}\mathscr{B}=v_\alpha\mu\mathrm{d}s\mathrm{d}x_3$) and of the top and bottom face areas $S^\pm$ (for which $\mathrm{d}S^\pm=\mu^\pm\mathrm{d}\sigma$), and applying Green's theorem wherever possible, considering the variations $\delta v_\alpha$, $\delta v_3$, $\delta\psi_\alpha$ and $\delta\psi_3$ as independent and arbitrary (except on $\mathscr{C}_v$ where these are prescribed), and in conjunction with Hamilton's condition according to which the virtual displacements vanish at $t_1$ and $t_2$, and after lengthy calculations one obtains the equations of motion/equilibrium and the boundary conditions. It clearly appears that by virtue of Eqs. (10) and of Eqs. (20) through Eq. (24) that have to be used in Eq. (48), Hamilton's principle will include the effects of interfacial displacement jumps connected with interfacial transverse stresses through Eqs. (18a) and (18b).

### 7.1. The equations of motion/equilibrium

The equations of motion are obtained in the form

$$\delta v_\alpha:\ \left.\overset{0}{L}{}^{\alpha\beta}\right|_\beta-b^\alpha_\beta Q^{\beta3}+\overset{0}{p}{}^\alpha+\overset{0}{F}{}^\alpha+\overset{0}{I}{}^\alpha=0, \tag{50a}$$

$$\delta v_3:\ Q^{\beta3}\big|_\beta+b_{\alpha\beta}\overset{0}{L}{}^{\alpha\beta}+\overset{0}{p}{}^3+\overset{0}{F}{}^3+\overset{0}{I}{}^3=0, \tag{50b}$$

$$\delta\psi_\alpha:\ \left.\overset{1}{L}{}^{\alpha\beta}\right|_\beta-Q^{\alpha3}+\overset{1}{p}{}^\alpha+\overset{1}{F}{}^\alpha+\overset{1}{I}{}^\alpha=0, \tag{50c}$$

$$\delta\psi_3:\ S^{\beta3}\big|_\beta-V^{33}+b_{\alpha\beta}P^{\alpha\beta}+\overset{1}{p}{}^3+\overset{1}{F}{}^3+\overset{1}{I}{}^3=0. \tag{50d}$$

The equations of motion as expressed by Eqs. (50a)–(50d) have to be supplemented by that expressing the symmetry of the stress tensor. Following the procedure developed e.g. in Librescu (1975a,b), the derivation of this additional equation is straightforward and is given by

$$\epsilon_{\lambda\mu}\left(\overset{0}{L}{}^{\mu\lambda}-b^\mu_\beta\overset{1}{L}{}^{\beta\lambda}\right)=0. \tag{50e}$$

where $\varepsilon_{\lambda\mu}$ is the antisymmetric surface tensor. In the case of the shallow-shell theory, this equation becomes an identity, and consequently it should be discarded.

The expressions of the gross stress resultants and stress couples are displayed in Appendix A. These equations are represented in terms of their counterparts associated with each constituent layer, defined as

$$
{}^{(k)}\overset{n}{L}{}^{\alpha\beta} = \int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+} \mu\sigma^{\alpha\sigma}\mu_\sigma^\beta\,(x^3)^n\mathrm{d}x^3, \qquad {}^{(k)}\overset{n}{L}{}^{\alpha 3} = \int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+} \mu\sigma^{\alpha 3}(x^3)^n\mathrm{d}x^3, \qquad n = 0, 1;
$$
$$
{}^{(k)}\overset{0}{L}{}^{33} = \int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+} \mu\sigma^{33}\mathrm{d}x^3. \tag{51a–c}
$$

### 7.2. Load and load couple

From the Hamilton variational principle, the load and load couples measured per unit area of $\sigma$ result as

$$
\begin{aligned}
\overset{0}{p}{}^\alpha &= \left[\mu\mu_\rho^\alpha\sigma^{3\rho}\right]_{S^-}^{S^+} + b_\beta^\alpha\left[\mu\mu_\rho^\gamma\sigma^{3\rho}\right]_{S^+}\sum_{\ell=1}^{N-1}\left[\left({}^{(N)}Z^+ - {}^{(\ell)}Z^+\right){}^{(\ell)}M_{\cdot\gamma}^\beta + {}^{(\ell)}\Psi_{\cdot\gamma}^\beta\right],\\
\overset{1}{p}{}^\alpha &= \left[\mu\mu_\rho^\alpha\sigma^{3\rho}x^3\right]_{S^-}^{S^+} + \left[\mu\mu_\rho^\gamma\sigma^{3\rho}\right]_{S^+}\sum_{\ell=1}^{N-1}\left[\left({}^{(N)}Z^+ - {}^{(\ell)}Z^+\right){}^{(\ell)}M_{\cdot\gamma}^\alpha + {}^{(\ell)}\Psi_{\cdot\gamma}^\alpha\right],\\
\overset{0}{p}{}^3 &= [\mu\sigma^{33}]_{S^-}^{S^+} - \left\{\left[\mu\mu_\rho^\gamma\sigma^{3\rho}\right]_{S^+}\sum_{\ell=1}^{N-1}\left[\left({}^{(N)}Z^+ - {}^{(\ell)}Z^+\right){}^{(\ell)}M_{\cdot\gamma}^\beta + {}^{(\ell)}\Psi_{\cdot\gamma}^\beta\right]\right\}\Bigg|_\beta,\\
\overset{1}{p}{}^3 &= [\mu\sigma^{33}\ x^3]_{S^-}^{S^+} - \left\{\left[\mu\mu_\rho^\gamma\sigma^{3\rho}\right]_{S^+}\sum_{\ell=1}^{N-1}\left[\left({}^{(N)}Z^+ - {}^{(\ell)}Z^+\right){}^{(\ell)}N_{\cdot\gamma}^\beta + {}^{(\ell)}Z^{+(\ell)}\Psi_{\cdot\gamma}^\beta + {}^{(\ell)}\phi_{\cdot\gamma}^\beta\right]\right\}\Bigg|_\beta\\
&\quad + [\mu\sigma^{33}]_{S^+}\sum_{\ell=1}^{N-1}\left[\left({}^{(N)}Z^+ - {}^{(\ell)}Z^+\right) = {}^{(\ell)}\Lambda_{\cdot 3}^3 + {}^{(\ell)}\Psi_{\cdot 3}^3\right].
\end{aligned} \tag{52a–d}
$$

Herein, $(\overset{0}{p}{}^\alpha, \overset{0}{p}{}^3)$ and $\overset{1}{p}{}^\alpha$ denote the external surface loads and load couple components, respectively, and $[\cdot]_{S^-}^{S^+} \equiv [\cdot]_{S^+} - [\cdot]_{S^-}$.

### 7.3. Body and inertia couples

From Hamilton's variational principle, one also obtains the expressions of the body gross force and body couple resultants as

$$
\begin{aligned}
\overset{0}{F}{}^\alpha &= \sum_{k=1}^{N}\left\{{}^{(k)}\overset{0}{F}{}^\alpha + b_\beta^\alpha\sum_{\ell=1}^{k-1}\left[\left({}^{(k)}\overset{1}{F}{}^\lambda - {}^{(\ell)}Z^{+(k)}\overset{0}{F}{}^\lambda\right){}^{(\ell)}M_{\cdot\lambda}^\beta + {}^{(k)}\overset{0}{F}{}^{\lambda(\ell)}\Psi_{\cdot\lambda}^\beta\right]\right\},\\
\overset{1}{F}{}^\alpha &= \sum_{k=1}^{N}\left\{{}^{(k)}\overset{1}{F}{}^\alpha + \sum_{\ell=1}^{k-1}\left[\left({}^{(k)}\overset{1}{F}{}^\lambda - {}^{(\ell)}Z^{+(k)}\overset{0}{F}{}^\lambda\right){}^{(\ell)}M_{\cdot\lambda}^\alpha + {}^{(k)}\overset{0}{F}{}^{\lambda(\ell)}\Psi_{\cdot\lambda}^\alpha\right]\right\},\\
\overset{0}{F}{}^3 &= \sum_{k=1}^{N}\left\{{}^{(k)}\overset{0}{F}{}^3 - \sum_{\ell=1}^{k-1}\left[\left({}^{(k)}\overset{1}{F}{}^\lambda - {}^{(\ell)}Z^{+(k)}\overset{0}{F}{}^\lambda\right){}^{(\ell)}M_{\cdot\lambda}^\beta + {}^{(k)}\overset{0}{F}{}^{\lambda(\ell)}\Psi_{\cdot\lambda}^\beta\right]\Bigg|_\beta\right\},\\
\overset{1}{F}{}^3 &= \sum_{k=1}^{N}\Bigg\{{}^{(k)}\overset{1}{F}{}^3 + \sum_{\ell=1}^{k-1}\Bigg[\left({}^{(k)}\overset{1}{F}{}^3 - {}^{(\ell)}Z^{+(k)}\overset{0}{F}{}^3\right){}^{(\ell)}\Lambda_{\cdot 3}^3 + {}^{(k)}\overset{0}{F}{}^{3(\ell)}\Psi_{\cdot 3}^3\\
&\quad - \left[\left({}^{(k)}\overset{1}{F}{}^\alpha - {}^{(\ell)}Z^{+(k)}\overset{0}{F}{}^\alpha\right){}^{(\ell)}N_{\cdot\alpha}^\beta + {}^{(k)}\overset{0}{F}{}^\lambda\left({}^{(\ell)}Z^{+(\ell)}\Psi_{\cdot\lambda}^\beta + {}^{(\ell)}\phi_{\cdot\lambda}^\beta\right)\right]\Bigg|_\beta\Bigg]\Bigg\}.
\end{aligned} \tag{53a–d}
$$

Herein, $(\overset{0}{F}{}^{\alpha}, \overset{0}{F}{}^{3})$ and $\overset{1}{F}{}^{\alpha}$ denote the gross body forces and body couples, respectively, measured per unit area of the undeformed reference surface, while

$$ {}^{(k)}\overset{n}{F}{}^{\alpha} = \int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+} \rho_0 \mu \mu_{\lambda}^{\alpha} H^{\lambda} (x^3)^n \mathrm{d}x^3 \quad \text{and} \quad {}^{(k)}\overset{n}{F}{}^{3} = \int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+} \rho_0 \mu \, H^3 (x^3)^n \mathrm{d}x^3. \tag{54a, b} $$

Assuming that the bond is without inertia, (Jones and Whittier, 1967), the gross inertia forces and inertia couples $\overset{0}{I}{}^{\alpha}$, $\overset{1}{I}{}^{\alpha}$, $\overset{0}{I}{}^{3}$ and $\overset{1}{I}{}^{3}$ can be obtained from Eqs. (53a–d) by replacing ${}^{(k)}\overset{n}{F}{}^{\alpha}$, ${}^{(k)}\overset{n}{F}{}^{3}$ by ${}^{(k)}\overset{n}{I}{}^{\alpha}$, ${}^{(k)}\overset{n}{I}{}^{3}$ $(n = 0, 1)$, respectively. The latter quantities, representing the inertia terms per each lamina are expressible by replacing in Eq. (54a,b) $H^{\lambda}$ and $H^3$ by $-\rho_0 \ddot{V}^{\lambda}$ and $-\rho_0 \ddot{V}^3$, respectively.

## 7.4. Boundary conditions

From the line integral arising in Eq. (48), using the relationship

$$ v_{3|\alpha} \equiv v_{3,\alpha} = \frac{\partial v_3}{\partial v} v_{\alpha} + \frac{\partial v_3}{\partial s} \tau_{\alpha}, \tag{55} $$

where $\partial/\partial v$ and $\partial/\partial s$ denote, respectively, the partial derivatives along the normal and tangent to $\mathscr{C}$, one obtains the static and geometric boundary conditions on $\mathscr{C}_f$ and $\mathscr{C}_v$, $(\mathscr{C}_f \cap \mathscr{C}_v = \emptyset)$.

The boundary conditions on $\mathscr{C}_f$ result as

$$ \begin{aligned}
&\delta v_v: \ \overset{0}{L}{}^{\alpha\beta} v_{\alpha} v_{\beta} + \tau_t K^{\alpha\beta} \tau_{\alpha} v_{\beta} = \underset{\sim}{\overset{0}{L}}{}^{\alpha\beta} v_{\alpha} v_{\beta} + \tau_t \underset{\sim}{K}{}^{\alpha\beta} \tau_{\alpha} v_{\beta}, \\
&\delta v_\tau: \ \overset{0}{L}{}^{\alpha\beta} \tau_{\alpha} v_{\beta} - \sigma_t K^{\alpha\beta} \tau_{\alpha} v_{\beta} = \underset{\sim}{\overset{0}{L}}{}^{\alpha\beta} \tau_{\alpha} v_{\beta} - \sigma_t \underset{\sim}{K}{}^{\alpha\beta} \tau_{\alpha} v_{\beta}, \\
&\delta v_3: \ \left(Q^{\alpha 3} - G^{\alpha} + J^{\alpha}\right) v_{\alpha} + \frac{\partial}{\partial s}\left(K^{\alpha\beta} \tau_{\alpha} v_{\beta}\right) = \left(\underset{\sim}{\overset{0}{Q}}{}^{\alpha 3} + P^{\alpha}\right) v_{\alpha} + \frac{\partial}{\partial s}\left(\underset{\sim}{K}{}^{\alpha\beta} \tau_{\alpha} v_{\beta}\right), \\
&\delta \phi_v: \ K^{\alpha\beta} v_{\alpha} v_{\beta} = \underset{\sim}{K}{}^{\alpha\beta} v_{\alpha} v_{\beta}, \\
&\delta \psi_v: \ \left(\overset{1}{L}{}^{\alpha\beta} - K^{\alpha\beta}\right) v_{\alpha} v_{\beta} = \left(\underset{\sim}{\overset{1}{L}}{}^{\alpha\beta} - \underset{\sim}{K}{}^{\alpha\beta}\right) v_{\alpha} v_{\beta}, \\
&\delta \psi_\tau: \ \left(\overset{1}{L}{}^{\alpha\beta} - K^{\alpha\beta}\right) \tau_{\alpha} v_{\beta} = \left(\underset{\sim}{\overset{1}{L}}{}^{\alpha\beta} - \underset{\sim}{K}{}^{\alpha\beta}\right) \tau_{\alpha} v_{\beta}, \\
&\delta \psi_3: \ \left(S^{\beta 3} - H^{\beta} + K^{\beta}\right) v_{\beta} + \frac{\partial}{\partial s}\left(R^{\alpha\beta} \tau_{\alpha} v_{\beta}\right) = \left(\underset{\sim}{\overset{1}{Q}}{}^{\beta 3} - \underset{\sim}{S}{}^{\beta} + \underset{\sim}{T}{}^{\beta}\right) v_{\beta} + \frac{\partial}{\partial s}\left(\underset{\sim}{R}{}^{\alpha\beta} \tau_{\alpha} v_{\beta}\right), \\
&\delta \psi_{3,v}: \ R^{\alpha\beta} v_{\alpha} v_{\beta} = \underset{\sim}{R}{}^{\alpha\beta} v_{\alpha} v_{\beta}.
\end{aligned} \tag{56a–h} $$

The boundary conditions on $\mathscr{C}_v$ are

$$ \begin{aligned}
&v_v = \underset{\sim v}{v}, \quad v_\tau = \underset{\sim \tau}{v}, \\
&v_3 = \underset{\sim 3}{v}, \quad \phi_v = \underset{\sim v}{\phi}, \\
&\psi_v = \underset{\sim v}{\psi}, \quad \psi_\tau = \underset{\sim \tau}{\psi}, \\
&\phi_3 = \underset{\sim 3}{\phi}, \quad \phi_{3,v} = \underset{\sim 3,v}{\phi}.
\end{aligned} \tag{57a–h} $$

In Eqs. (56a–h) and (57a–h), the prescribed quantities are underscored by a wavy line, while $\tau_t (\equiv -b_{\alpha\beta} v^{\alpha} \tau^{\beta})$ and $\sigma_t (\equiv b_{\alpha\beta} \tau^{\alpha} \tau^{\beta})$ denote the geodesic torsion and normal curvature of $\mathscr{C}$, respectively, and

$$\phi_\nu = \phi_\alpha \nu^\alpha \quad \text{where} \quad \phi_\alpha = v_{3,\alpha} + b^\lambda_\alpha v_\lambda. \tag{58a, b}$$

The remaining notations are displayed in Appendix B.

At points on $\mathscr{C}$ located at $s = s_i, i = \overline{1,p}$, at which $K^{\alpha\beta}$ and $R^{\alpha\beta}$ have jump discontinuities induced by corners or concentrated loads, one obtains the conditions

$$\begin{aligned} &\delta v_3(s_i) \quad (K^{\alpha\beta}\tau_\alpha \nu_\beta)\Big|_{s_i-0}^{s_i+0} = (\underset{\sim}{K}^{\alpha\beta}\tau_\alpha \nu_\beta)\Big|_{s_i-0}^{s_i+0} \\ &\delta\psi_3(s_i) \quad (R^{\alpha\beta}\tau_\sigma \nu_\beta)\Big|_{s_i-0}^{s_i+0} = (\underset{\sim}{R}^{\alpha\beta}\tau_\alpha \nu_\beta)\Big|_{s_i-0}^{s_i+0} \end{aligned}. \tag{59a, b}$$

Consistent with the number of eight boundary conditions, which have to be prescribed at each edge of the shell, the associated system of governing equations is of the order of 16, that is, six orders higher than the *standard* first-order transverse shear deformation theory, and four orders higher as compared to the case of laminated shells incorporating sliding bonding imperfections, where the assumption of transverse normal incompressibility is adopted. This means that incorporation of full interface bonding imperfections and transverse normal compressibility is paid by a substantial increase of the order of the governing equations.

It should be remarked that along a clamped edge requiring fulfillment of homogeneous kinematical conditions (57a–h), the sliding effect is completely eliminated.

## 8. Constitutive equations

The material of each constituent layer is assumed homogeneous and anisotropic, the anisotropy being of the symmetry type with respect to the surface $x^3 = 0$ (monoclinic symmetry). For the actual fibrous reinforced composite structures, the constituent orthotropic materials whose principal axes are not aligned with the structure axes, feature monoclinic-type anisotropy (Librescu, 1975a,b).

For this case of anisotropy, the constitutive equations for the $k$th layer assume the form

$${}^{(k)}\sigma^{\alpha\beta} = {}^{(k)}E^{\alpha\beta\omega\rho}E^{(k)}_{\omega\rho} + {}^{(k)}E^{\alpha\beta 33}E^{(k)}_{33}, \tag{60a}$$

$${}^{(k)}\sigma^{\alpha 3} = 2\,{}^{(k)}E^{\alpha 3\omega 3}E^{(k)}_{\omega 3}, \tag{60b}$$

$${}^{(k)}\sigma^{33} = {}^{(k)}E^{3333}E^{(k)}_{33} + {}^{(k)}E^{33\alpha\beta}E^{(k)}_{\alpha\beta}. \tag{60c}$$

Having in view that the tensors of elastic moduli $E^{\alpha\beta\gamma\delta}, E^{\alpha\beta 33}, E^{\alpha 3\beta 3}$ are space tensors, these are expressible in terms of their surface counterparts as

$$E^{\alpha\beta\gamma\delta} = (\mu^{-1})^\alpha_\nu \, (\mu^{-1})^\beta_\rho \, (\mu^{-1})^\gamma_\omega \, (\mu^{-1})^\delta_\eta \overline{E}^{\nu\rho\omega\eta}, \tag{61a}$$

$$E^{\alpha 3\beta 3} = (\mu^{-1})^\alpha_\nu \, (\mu^{-1})^\beta_\rho \, \overline{E}^{\nu 3\rho 3}, \tag{61b}$$

$$E^{\omega\alpha 33} = (\mu^{-1})^\omega_\phi \, (\mu^{-1})^\alpha_\gamma \, \overline{E}^{\phi\gamma 33}, \tag{61c}$$

$$E^{3333} = \overline{E}^{3333}. \tag{61d}$$

At this point, it should be remarked that by virtue of conditions proper to shallow shells (Eqs. (29) and (61a)–(61d)), the space tensors of elastic moduli $E^{ijmn}$ coincide with their surface counterparts $\bar{E}^{ijmn}$.

By virtue of Eqs. (21a)–(21c) through (24) expressing the 3-D strains, and of those for the stress resultants and stress couples, Eq. (51a)–(51c), one obtains

$$ {}^{(k)}\overset{0}{L}{}^{\alpha\beta} = {}^{(k)}{}_{0}B^{\alpha\beta\omega\rho}\overset{0}{e}{}^{(k)}_{\omega\rho} + {}^{(k)}{}_{1}B^{\alpha\beta\omega\rho}\overset{1}{e}{}^{(k)}_{\omega\rho} + {}^{(k)}{}_{2}B^{\alpha\beta\omega\rho}\ \overset{2}{e}{}^{(k)}_{\omega\rho} + {}^{(k)}{}_{0}B^{\alpha\beta 33}\overset{0}{e}{}^{(k)}_{33}, \tag{62a} $$

$$ {}^{(k)}\overset{1}{L}{}^{\alpha\beta} = {}^{(k)}{}_{1}B^{\alpha\beta\omega\rho}\overset{0}{e}{}^{(k)}_{\omega\rho} + {}^{(k)}{}_{2}B^{\alpha\beta\omega\rho}\overset{1}{e}{}^{(k)}_{\omega\rho} + {}^{(k)}{}_{3}B^{\alpha\beta\omega\rho}\ \overset{2}{e}{}^{(k)}_{\omega\rho} + {}^{(k)}{}_{1}B^{\alpha\beta 33}\overset{0}{e}{}^{(k)}_{33}, \tag{62b} $$

$$ {}^{(k)}\overset{0}{L}{}^{\alpha 3} = {}^{(k)}{}_{0}B^{\alpha 3\omega 3}\overset{0}{e}{}^{(k)}_{\omega 3} + {}^{(k)}{}_{1}B^{\alpha 3\omega 3}\overset{1}{e}{}^{(k)}_{\omega 3}, \tag{62c} $$

$$ {}^{(k)}\overset{1}{L}{}^{\alpha 3} = {}^{(k)}{}_{1}B^{\alpha 3\omega 3}\overset{0}{e}{}^{(k)}_{\omega 3} + {}^{(k)}{}_{2}B^{\alpha 3\omega 3}\overset{1}{e}{}^{(k)}_{\omega 3}, \tag{62d} $$

$$ {}^{(k)}\overset{0}{L}{}^{33} = {}^{(k)}{}_{0}B^{3333}\ \overset{0}{e}{}^{(k)}_{33} + {}^{(k)}{}_{0}B^{33\alpha\beta}\overset{0}{e}{}^{(k)}_{\alpha\beta} + {}^{(k)}{}_{1}B^{33\alpha\beta}\overset{1}{e}{}^{(k)}_{\alpha\beta} + {}^{(k)}{}_{2}B^{33\alpha\beta}\overset{2}{e}{}^{(k)}_{\alpha\beta}. \tag{62e} $$

The stiffness quantities appearing in Eqs. (62a)–(62e) are expressed as

$$ {}^{(k)}{}_{n}B^{\alpha\beta\omega\pi} = {}^{(k)}\overline{E}^{\alpha\sigma\lambda\rho} \int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+} \mu(\mu^{-1})^{\beta}_{\sigma}\ (\mu^{-1})^{\omega}_{\lambda}\ (\mu^{-1})^{\pi}_{\rho}\ (x^3)^n\,\mathrm{d}x^3, \tag{63a} $$

$$ {}^{(k)}{}_{n}B^{\alpha 3\beta 3} = {}^{(k)}\overline{E}^{\gamma 3\sigma 3} \int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+} \mu(\mu^{-1})^{\alpha}_{\gamma}\ (\mu^{-1})^{\beta}_{\sigma}\ (x^3)^n\,\mathrm{d}x^3, \tag{63b} $$

$$ {}^{(k)}{}_{n}B^{\omega\sigma 33} = {}^{(k)}\overline{E}^{\pi\gamma 33} \int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+} \mu(\mu^{-1})^{\omega}_{\pi}\ (\mu^{-1})^{\sigma}_{\gamma}\ (x^3)^n\,\mathrm{d}x^3, \tag{63c} $$

$$ {}^{(k)}{}_{0}E^{3333} = {}^{(k)}\overline{E}^{3333}\left({}^{(k)}Z^+ - {}^{(k)}Z^-\right). \tag{63d} $$

Using the expression of the inverse shifter tensor, Eq. (4), more elaborated representations for the laminae stiffness tensors, Eqs. (63a)–(63d), can be found in Librescu (1975a,b). Based on these expressions, non-contradictory variants of the constitutive equations fulfilling identically the supplementary equilibrium equation (50e) can be obtained.

For shear deformable and unshearable laminated composite shells, such constitutive equations have been obtained by Librescu (1975a,b, 1976), respectively, whereas for single-layered shearable shells, by Naghdi (1963).

For *shallow*-shell theory, the stiffness quantities can be expressed as

$$ \begin{aligned} {}^{(k)}{}_{n}B^{\alpha\beta\omega\rho} &= {}^{(k)}E^{\alpha\beta\omega\rho}\ \eta^{(k)}(n+1), \\ {}^{(k)}{}_{n}B^{\omega\sigma 33} &= {}^{(k)}{}_{n}B^{33\omega\sigma} = {}^{(k)}E^{\omega\sigma 33}\eta^{(k)}(n+1), \\ {}^{(k)}{}_{n}B^{\omega 3\sigma 3} &= {}^{(k)}E^{\omega 3\sigma 3}\eta^{(k)}(n+1), \end{aligned} \tag{64a–c} $$

whereas Eq. (63d) remains unchanged.

Herein,

$$\eta^{(k)}(Q) = \frac{1}{Q}\left(\left({}^{(k)}Z^{+}\right)^{Q} - \left({}^{(k)}Z^{-}\right)^{Q}\right). \tag{65}$$

In the case of shells composed of $N = 2m + 1$ layers symmetrically distributed about the global middle surface (considered to coincide with the reference surface), care should be exercised when determining the gross stress resultants defined in Appendix A as well as $(\overset{n}{L}{}^{\alpha\beta}; \overset{n}{L}{}^{\alpha 3}; \overset{n}{L}{}^{33}) \equiv \sum_{k=1}^{N} ({}^{(k)}\overset{n}{L}{}^{\alpha\beta};\ {}^{(k)}\overset{n}{L}{}^{\alpha 3};\ {}^{(k)}\overset{n}{L}{}^{33})$ that appear in different contexts in Eqs. (A.1)–(A.7). In such a case, the global stiffness quantities can be defined as

$${}_{n}\mathscr{B}^{\alpha\beta\omega\rho} = \sum_{k=1}^{m+1} {}^{(k)}E^{\alpha\beta\omega\rho}\eta^{(k)}(n+1), \tag{66a}$$

$${}_{n}\mathscr{B}^{\alpha 3\omega 3} = \sum_{k=1}^{m+1} {}^{(k)}E^{\alpha 3\omega 3}\eta^{(k)}(n+1), \tag{66b}$$

$${}_{n}\mathscr{B}^{\omega\sigma 33} = \sum_{k=1}^{m+1} {}^{(k)}E^{\omega\sigma 33}\eta^{(k)}(n+1), \tag{66c}$$

where for symmetrically laminated shells

$$\eta^{(k)}(Q) = \begin{cases} \frac{2}{Q}\left[\left({}^{(k)}Z^{+}\right)^{Q} - \left({}^{(k)}Z^{-}\right)^{Q}\right] & \text{for odd } Q, \\ 0 & \text{for even } Q. \end{cases} \tag{67}$$

For this case, ${}_{2s+1}\mathscr{B}^{\alpha\beta\omega\rho} = 0$, ${}_{2s+1}\mathscr{B}^{\alpha 3\omega 3} = 0$, ${}_{2s+1}\mathscr{B}^{\omega\sigma 33} = 0$, whereas ${}_{2s}\mathscr{B}^{\alpha\beta\omega\rho}$, ${}_{2s}\mathscr{B}^{\alpha 3\omega 3}$, ${}_{2s}\mathscr{B}^{\omega\sigma 33} (s = 0, 1, \ldots)$ remain different from zero.

The former stiffness quantities in the case of symmetrically laminated shells become immaterial and are customarily referred to as the bending–stretching coupling stiffnesses. However, as is clearly seen, here these have a more general meaning than in its classical context.

Moreover, in the present context, the symmetry in geometry and material properties should be complemented by that involving the interfacial flaws also. As a result, in this case, for the determination of the gross stress resultants and stress couples as defined by Eqs. (A.1)–(A.7), in addition to the previously indicated considerations regarding the vanishing of bending–stretching stiffness quantities, that related to the symmetry of interfacial flaws also has to be addressed. Only in the framework of such a *generalized symmetry*, the theory of laminated composite *flat structures* featuring interlaminar bonding imperfections will experience the decoupling of bending and stretching states of stress.

## 9. Discussion

In this paper, a general theory of anisotropic laminated shells incorporating the effects of the damaged bonding imperfections has been presented. Herein, laminated composite shells with imperfect bonded interfaces include as special cases those featuring perfectly bonded interfaces, perfect slip interfaces, and complete debonding implying stress free interfaces. The equations are derived for the general case of deep shells. In addition, special cases involving the theory of shallow shells are also considered. It was shown that in all these cases, the theory can be formulated in terms of six 2-D functions, namely $v_\alpha$, $\psi_\alpha$, $v_3$ and $\psi_3$. This developed theory has the potential to provide valuable information on the implications of bonding defects on the load carrying capacity of laminated shells. Moreover, using the directionality property provided by advanced fiber reinforced composite materials, ways of mitigating the damaging effects of bonding imperfections can be devised.

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## Appendix A

The expressions of the gross stress resultants and stress couples appearing in Eqs. (50a)–(50e).

$$\overset{n}{L}{}^{\alpha\beta} = \sum_{k=1}^{N} {}^{(k)}\overset{n}{L}{}^{\alpha\beta}, \qquad n = 0, 1 \tag{A.1}$$

$$Q^{\beta 3} = \sum_{k=1}^{N} \left\{ {}^{(k)}\overset{0}{L}{}^{\lambda 3}\left(\delta^{\beta}_{\lambda} + {}^{(k)}P^{\beta}_{\cdot\lambda}\right) + {}^{(k)}\overset{0}{L}{}^{\omega\sigma}|_{\sigma} \sum_{\ell=1}^{k-1}\left({}^{(\ell)}Z^{+(\ell)}M^{\beta}_{\cdot\omega} - {}^{(\ell)}\Psi^{\beta}_{\cdot\omega}\right) - {}^{(k)}\overset{1}{L}{}^{\omega\sigma}|_{\sigma} \sum_{\ell=1}^{k-1} {}^{(\ell)}M^{\beta}_{\cdot\omega}, \right. \tag{A.2}$$

$$\begin{aligned} S^{\beta 3} = \sum_{k=1}^{N} \Bigg\{ & {}^{(k)}\overset{0}{L}{}^{\lambda 3(k)}F^{\beta}_{\cdot\lambda} + {}^{(k)}\overset{1}{L}{}^{\beta 3}\left(1 + {}^{(k)}P^{3}_{\cdot 3}\right) + {}^{(k)}\overset{(0)}{L}{}^{\omega\rho}|_{\rho} \sum_{\ell=1}^{k-1}\left[{}^{(\ell)}Z^{+}\left({}^{(\ell)}N^{\beta}_{\cdot\omega} - {}^{(\ell)}\Psi^{\beta}_{\cdot\omega}\right) - {}^{(\ell)}\phi^{\beta}_{\cdot\omega}\right] \\ & - {}^{(k)}\overset{1}{L}{}^{\omega\rho}|_{\rho} \sum_{\ell=1}^{k-1} {}^{(\ell)}N^{\beta}_{\cdot\omega} \Bigg\}, \end{aligned} \tag{A.3}$$

$$P^{\alpha\beta} = \sum_{k=1}^{N} \left\{ {}^{(k)}\overset{1}{L}{}^{\alpha\beta} + {}^{(k)}\overset{0}{L}{}^{\alpha\beta} \sum_{\ell=1}^{k-1} \Psi^{3}_{\cdot 3} + \sum_{\ell=1}^{k-1}\left({}^{(k)}\overset{1}{L}{}^{\alpha\beta} - {}^{(\ell)}Z^{+(k)}\overset{0}{L}{}^{\alpha\beta}\right){}^{(\ell)}\Lambda^{3}_{\cdot 3} \right\}, \tag{A.4}$$

$$V^{33} = \sum_{k=1}^{N} \left\{ {}^{(k)}\overset{0}{L}{}^{33}\left(1 + {}^{(k)}P^{3}_{\cdot 3}\right) \right\}, \tag{A.5}$$

where

$${}^{(k)}P^{\beta}_{\cdot\lambda} = \sum_{\ell=1}^{k-1} {}^{(1;\ell)}\Lambda^{\beta}_{\cdot\lambda}, \qquad {}^{(k)}P^{3}_{\cdot 3} = \sum_{\ell=1}^{k-1} {}^{(\ell)}\Lambda^{3}_{\cdot 3}, \tag{A.6}$$

and

$${}^{(k)}F^{\omega}_{\cdot\alpha} = \sum_{\ell=1}^{k-1} \left\{ {}^{(1)}Z^{+(1;k)}\Lambda^{\omega}_{\cdot\alpha} + \sum_{m=2}^{\ell}\left[\left({}^{(m)}Z^{+} - {}^{(m-1)}Z^{+}\right)\left(1 + {}^{(m)}P^{3}_{\cdot 3}\right){}^{(m;\ell)}\Lambda^{\omega}_{\cdot\alpha}\right] - {}^{(\ell)}Z^{+(\ell)}\Lambda^{3}_{\cdot 3}\delta^{\omega}_{\alpha} \right\}. \tag{A.7}$$

**Appendix B.**

Quantities appearing in the boundary conditions, (56a)–(56h).

$$K^{\alpha\beta} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\left[\left({}^{(\ell)}Z^{+(k)}\overset{0}{L}{}^{\lambda\beta} - {}^{(k)}\overset{1}{L}{}^{\lambda\beta}\right){}^{(\ell)}M^{\alpha}_{\cdot\lambda} - {}^{(k)}\overset{0}{L}{}^{\lambda\beta(\ell)}\Psi^{\alpha}_{\cdot\lambda}\right],$$

$$R^{\alpha\beta} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\left[\left({}^{(\ell)}Z^{+(k)}\overset{0}{L}{}^{\lambda\beta} - {}^{(k)}\overset{1}{L}{}^{\lambda\beta}\right){}^{(\ell)}N^{\alpha}_{\cdot\lambda} - {}^{(k)}\overset{0}{L}{}^{\lambda\beta}\left({}^{(\ell)}Z^{+(\ell)}\Psi^{\alpha}_{\cdot\lambda} + {}^{(\ell)}\phi^{\alpha}_{\cdot\lambda}\right)\right.,$$

$$G^{\alpha} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\left[\left(-{}^{(\ell)}Z^{+(k)}\overset{0}{F}{}^{\lambda} + {}^{(k)}\overset{1}{F}{}^{\lambda}\right){}^{(\ell)}M^{\alpha}_{\cdot\lambda} + {}^{(k)}\overset{0}{F}{}^{\lambda(\ell)}\Psi^{\alpha}_{\cdot\lambda}\right],$$

$$J^{\alpha} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\left[\left(-{}^{(\ell)}Z^{+(k)}\overset{0}{I}{}^{\lambda} + {}^{(k)}\overset{1}{I}{}^{\lambda}\right){}^{(\ell)}M^{\alpha}_{\cdot\lambda} + {}^{(k)}\overset{0}{I}{}^{\lambda(\ell)}\Psi^{\alpha}_{\cdot\lambda}\right],$$

$$H^{\alpha} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\left[\left(-{}^{(\ell)}Z^{+(k)}\overset{0}{F}{}^{\lambda} + {}^{(k)}\overset{1}{F}{}^{\lambda}\right){}^{(\ell)}N^{\alpha}_{\cdot 3} + {}^{(k)}\overset{0}{F}{}^{\lambda}\left({}^{(\ell)}Z^{+(\ell)}\Psi^{\alpha}_{\cdot\lambda} + {}^{(\ell)}\phi^{\alpha}_{\cdot\lambda}\right)\right],$$

$$K^{\alpha} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\left[\left(-{}^{(\ell)}Z^{+(k)}\overset{0}{I}{}^{\lambda} + {}^{(k)}\overset{1}{I}{}^{\lambda}\right){}^{(\ell)}N^{\alpha}_{\cdot\lambda} + {}^{(k)}\overset{0}{I}{}^{\lambda}\left({}^{(\ell)}Z^{+(\ell)}\Psi^{\alpha}_{\cdot\lambda} + {}^{(\ell)}\phi^{\alpha}_{\cdot\lambda}\right)\right],$$

$$\underset{\sim}{\overset{0}{L}}{}^{\alpha\beta} = \sum_{k=1}^{N}(k)\underset{\sim}{\overset{0}{L}}{}^{\alpha\beta}; \qquad \underset{\sim}{\overset{1}{L}}{}^{\alpha\beta} = \sum_{k=1}^{N}(k)\underset{\sim}{\overset{1}{L}}{}^{\alpha\beta}; \qquad \underset{\sim}{\overset{0}{Q}}{}^{\alpha 3} = \sum_{k=1}^{N}(k)\underset{\sim}{\overset{0}{L}}{}^{\alpha 3}, \qquad \underset{\sim}{\overset{1}{Q}}{}^{\alpha 3} = \sum_{k=1}^{N}(k)\underset{\sim}{\overset{1}{L}}{}^{\alpha 3},$$

$${}^{(k)}\underset{\sim}{\overset{n}{L}}{}^{\alpha\beta} = \int_{{}^{(k)}Z^{-}}^{{}^{(k)}Z^{+}} \mu\mu^{\alpha}_{\lambda}\underset{\sim}{\sigma}{}^{\beta\lambda}(x^{3})^{n}\,\mathrm{d}x_{3}, \qquad n = 0, 1,$$

$${}^{(k)}\underset{\sim}{\overset{n}{L}}{}^{\beta 3} = \int_{{}^{(k)}Z^{-}}^{{}^{(k)}Z^{+}} \mu\underset{\sim}{\sigma}{}^{\beta 3}(x^{3})^{n}\,\mathrm{d}x_{3},$$

$$\underset{\sim}{K}{}^{\alpha\beta} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\left[\left({}^{(\ell)}Z^{+(k)}\underset{\sim}{\overset{0}{L}}{}^{\lambda\beta} - {}^{(k)}\underset{\sim}{\overset{1}{L}}{}^{\lambda\beta}\right){}^{(\ell)}M^{\alpha}_{\cdot\lambda} - {}^{(k)}\underset{\sim}{L}{}^{\lambda\beta(\ell)}\Psi^{\alpha}_{\cdot\lambda},\right],$$

$$\underset{\sim}{R}{}^{\alpha\beta} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\left[\left({}^{(\ell)}Z^{+(k)}\underset{\sim}{\overset{0}{L}}{}^{\lambda\beta} - {}^{(k)}\underset{\sim}{\overset{1}{L}}{}^{\lambda\beta}\right){}^{(\ell)}N^{\alpha}_{\cdot\lambda} - {}^{(k)}\underset{\sim}{\overset{0}{L}}{}^{\lambda\beta}\left({}^{(\ell)}Z^{+(\ell)}\Psi^{\alpha}_{\cdot\lambda} + {}^{(\ell)}\phi^{\alpha}_{\cdot\lambda}\right)\right],$$

$$\underset{\sim}{P}{}^{\alpha} = \left[\mu\mu^{\lambda}_{\rho}\,\underset{\sim}{\sigma}{}^{3\rho}\right]\Big|_{S^{+}}\sum_{\ell=1}^{k-1}\left[\left({}^{(N)}Z^{+} - {}^{(\ell)}Z^{+}\right){}^{(\ell)}M^{\alpha}_{\cdot\lambda} + {}^{(\ell)}\Psi^{\alpha}_{\cdot\lambda}\right],$$

$$\underset{\sim}{S}{}^{\alpha} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\left[\left({}^{(\ell)}Z^{+(k)}\underset{\sim}{\overset{0}{L}}{}^{3\alpha} - {}^{(k)}\underset{\sim}{\overset{1}{L}}{}^{3\alpha}\right){}^{(\ell)}\Lambda^{3}_{\cdot 3} - {}^{(k)}\underset{\sim}{\overset{0}{L}}{}^{3\alpha}\Psi^{3}_{\cdot 3}\right],$$

$$\underset{\sim}{T}{}^{\alpha} = \left(\mu\mu\overset{\lambda}{\omega}\underset{\sim}{\sigma}{}^{3\omega}\right)\Big|_{S^{+}}\sum_{\ell=1}^{N=1}\left[\left({}^{(N)}Z^{+} - {}^{(\ell)}Z^{+}\right){}^{(\ell)}N^{\alpha}_{\cdot\lambda} + {}^{(\ell)}Z^{+(\ell)}\Psi^{\alpha}_{\cdot\lambda} + {}^{(\ell)}\phi^{\alpha}_{\cdot\lambda})\right].$$

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